PERGAMON

International Journal of Solids and Structures 36 (1999) 1965–1983

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

# Bonded contact of a flexible elliptical disk with a transversely isotropic half-space

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Received 10 October 1997; in revised form 25 February 1998

**Abstract**

The article concerns the problem of bonded contact of a thin, flexible elliptical disk with a transversely isotropic half-space. Three different cases of loading have been considered: (a) the disk is loaded by a transverse force, whose line of action passes through the center of the disk and lies in the plane of the disk; (b) the disk is subjected to a rotation by a torque, whose axis is perpendicular to the surface of the half-space; (c) the half-space with the bonded disk is under uniform stress field at infinity. The problem corresponding to all three cases is reduced, in a unified manner, to a set of coupled two-dimensional integral equations. Closed-form solutions for these equations have been obtained by using Galin's theorem. © 1999 Elsevier Science Ltd. All rights reserved.

## 1. Introduction

While the contact problems of elastic stress distribution in isotropic materials have been investigated at great detail, relatively less work has been done on similar problems in anisotropic materials. This is primarily because of the greater difficulty of this type of problems involving more than two elastic constants. However, in the case of a transversely isotropic material whose constitutive behavior may be described by five independent elastic constants, solutions of a large number of problems can be found. Elliot (1948, 1949) seems to have first initiated work in this direction. In particular, Elliot (1949) investigated the axisymmetric problem of a transversely isotropic half-space indented by a rigid punch. Subsequently, Shield (1951) adapted Elliot's approach (Elliot, 1984) to solve a number of more difficult crack and punch problems for a transversely isotropic material, such as the problems of elliptical punch and crack. Sveklo (1964) employed the Smirnov–Sobolev technique to derive Boussinesq type solutions for a generally anisotropic half-space, in particular, a transversely isotropic half-space and used them to solve a number of axisymmetric and non-axisymmetric indentation problems for a transversely isotropic

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PII: S0020–7683(98)00069–9

solid (Sveklo, 1970). Willis (1966) extended Galin's theorem to solve the problem of Hertzian contact between two anisotropic solids. Conway et al. (1967a) were concerned with the problem of finding the location of maximum shearing stresses under a rigid sphere indenting a transversely isotropic half-space. These authors found that they would occur at a depth of $0.5a$ in contrast to the isotropic case (which is $0.47a$), where $a$ is the radius of the contact region. Later, Conway and Farnham (1967b) investigated the same issue for the case where the sphere is subjected to a transverse force. Dahan and Zarka (1977) investigated the axisymmetric problem of contact between a rigid sphere and a transversely isotropic half-space with resort to a Hankel-transform approach and presented extensive numerical results as to how the contact stress distributions are influenced by transverse isotropy. This problem was also considered by Guidera et al. (1978) and Pouyet and Lataillaze (1979). Borodachev (1990) used a variational approach to solve the problem of indentation of a transversely isotropic half-space by a rigid punch with a nearly circular base. Recently, Fabrikant (1997) obtained exact solution for the problem of contact interaction between circular punch and a transversely isotropic solid when tangential displacements are prescribed within the contact area and the rest of the surface is free. Readers interested in excellent reviews of the work in this area as well as in other mathematically similar areas of mixed boundary value problems of the elasticity theory are referred to the books by Galin (1976), Gladwell (1980) and Ting (1996).

The present article is concerned with the problem of contact between a tension-resistant, thin absolutely flexible disk of elliptical planform and a transversely isotropic half-space. Complete bonding is assumed to exist between them. Three different cases of loading have been considered: (a) the disk is loaded by a transverse force whose line of action passes through the center of the disk and lies in its plane; (b) the disk is twisted by a torque whose axis is perpendicular to the surface of the half-space; (c) the half-space with the bonded disk under uniform stress field at infinity in a plane parallel to the plane of the disk. By means of double Fourier transform, the problem for all three cases has been reduced, in a unified manner, to a set of coupled two-dimensional integral equations, exact solution of which has been derived by using Galin's theorem. The correctness of the solution has been checked against the solution of the corresponding problem for an isotropic half-space. To the best of our knowledge, the present solution is new.

The present article may be regarded as a sequel to the work by Alexandrov and Solov'ev (1966), who investigated the isotropic version of the problem. Also, the present problem is mathematically equivalent to that involving two transversely isotropic bodies already in Hertzian contact, in which an additional system of forces is applied to the bodies such that across the contact surface, one body exerts on the other a small additional transverse load and a couple. Viewed from this point, it is worth mentioning that the isotropic version of the problem investigated herein corresponding to the cases (a) and (b) were also addressed by Mindlin (1949) and Lure (1964) using a different approach.

We begin by introducing the notation which we shall make use of.

We define the two-dimensional Fourier transform of a function, $f(x, y)$ by the equation (Sneddon, 1972):

$$\tilde{f}(\alpha_1, \alpha_2) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\exp(ix\alpha_1 + iy\alpha_2)\,\mathrm{d}x\,\mathrm{d}y$$

and write $\tilde{f}(\alpha_1, \alpha_2) = \mathscr{F}[\{f(x, y); x \to \alpha_1\}, y \to \alpha_2]$. The inversion theorem for the Fourier operator $\mathscr{F}$ states that if $\tilde{f}$ is the Fourier transform of $f$, then

$$f(x, y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{f}(\alpha_1, \alpha_2)\exp(-ix\alpha_1 - iy\alpha_2)\,\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2$$

which we write as

$$f(x, y) = \mathscr{F}[\{\tilde{f}(\alpha_1, \alpha_2); \alpha_1 \to x\}, \alpha_2 \to y].$$

The basic results that we need are as follows:

$$\mathscr{F}\left[\left\{\frac{\partial f(x_1, x_2, z)}{\partial x_j}; x_1 \to \alpha_1\right\}; x_2 \to \alpha_2\right] = -i\alpha_j\tilde{f}(\alpha_1, \alpha_2, z), \quad j = 1, 2,$$

$$\mathscr{F}\left[\left\{\frac{\partial f(x_1, x_2, z)}{\partial z}; x_1 \to \alpha_1\right\}; x_2 \to \alpha\right] = \frac{\partial \tilde{f}(\alpha_1, \alpha_2, z)}{\partial z},$$

where $x_1 = x$ and $x_2 = y$.

We write convolution theorem in the form

$$\mathscr{F}^{-1}[\{\tilde{f}(\alpha_1, \alpha_2)\tilde{g}(\alpha_1, \alpha_2); \alpha_1 \to x\}; \alpha_2 \to y] = (f \circ g)(x, y),$$

where $(f \circ g)$ is defined by

$$(f \circ g)(x, y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x - \xi, y - \eta)g(\xi, \eta)\,\mathrm{d}\xi\,\mathrm{d}\eta.$$

## 2. Basic equations and potential solutions for transversely isotropic bodies

We consider a transversely isotropic solid occupying the half-space ($|x| < \infty, |y| < \infty, z \geqslant 0$), with the assumption that the axis of symmetry for the material is the $z$-axis. We denote the displacement vector at the point $(x, y, z)$ by $\hat{u}$ with the components $(u, v, w)$ and the stress tensor by $\hat{\sigma}$ with the components $\sigma_{xx}$, $\sigma_{yy}$, $\sigma_{zz}$, $\sigma_{yz}$, $\sigma_{zx}$, $\sigma_{xy}$. Then, the equilibrium of the solid is governed by the following equations:

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} = 0,$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} = 0,$$

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} = 0, \tag{1}$$

The stress–strain relationships for a transversely isotropic material are given by the following equations (Green and Zerna, 1968):

$$\sigma_{xx} = c_{11}\frac{\partial u}{\partial x} + c_{12}\frac{\partial v}{\partial y} + c_{13}\frac{\partial w}{\partial z},$$

$$\sigma_{yy} = c_{12}\frac{\partial u}{\partial x} + c_{11}\frac{\partial v}{\partial y} + c_{13}\frac{\partial w}{\partial z},$$

$$\sigma_{zz} = c_{13}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + c_{33}\frac{\partial w}{\partial z},$$

$$\sigma_{yz} = c_{44}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right),$$

$$\sigma_{zx} = c_{44}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right),$$

$$\sigma_{xy} = \frac{1}{2}(c_{11} - c_{12})\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right). \tag{2}$$

Elliot (1948) (see also Green and Zerna, 1968) showed that the equations of equilibrium (1) for a transversely isotropic, elastic solid can be expressed in terms of three potential functions, $\chi_\alpha$ ($\alpha = 1, 2, 3$), which obey the following Laplace-type equations:

$$\left(\nabla_1^2 + s_\alpha \frac{\partial^2}{\partial z^2}\right)\chi_\alpha = 0, \tag{3}$$

where

$$\nabla_1^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2},$$

$$s_3 = \frac{2c_{44}}{c_{11} - c_{12}}, \tag{4}$$

and $s_1$, $s_2$ are two distinct roots of the equation

$$c_{11}c_{44}s^2 + \{c_{13}(2c_{44} + c_{13}) - c_{11}c_{33}\}s + c_{33}c_{44} = 0. \tag{5}$$

In terms of $\chi_\alpha$, the components of displacements and stresses are given by the relations:

$$u = \frac{\partial}{\partial x}(\chi_1 + \chi_2) + \frac{\partial \chi_3}{\partial y},$$

$$v = \frac{\partial}{\partial y}(\chi_1 + \chi_2) - \frac{\partial \chi_3}{\partial x},$$

$$w = k_1\frac{\partial \chi_1}{\partial z} + k_2\frac{\partial \chi_2}{\partial z},$$

$$\sigma_{zz} = (k_1 c_{33} - s_1 c_{13})\frac{\partial^2 \chi_1}{\partial z^2} + (k_2 c_{33} - s_2 c_{13})\frac{\partial^2 \chi_2}{\partial z^2},$$

$$\sigma_{yz} = c_{44}\left\{(1+k_1)\frac{\partial^2\chi_1}{\partial y\,\partial z}+(1+k_2)\frac{\partial^2\chi_2}{\partial y\,\partial z}-\frac{\partial^2\chi_3}{\partial x\,\partial z}\right\},$$

$$\sigma_{zx} = c_{44}\left\{(1+k_1)\frac{\partial^2\chi_1}{\partial x\,\partial z}+(1+k_2)\frac{\partial^2\chi_2}{\partial x\,\partial z}+\frac{\partial^2\chi_3}{\partial y\,\partial z}\right\}, \tag{6}$$

where

$$k_\alpha = \frac{c_{11}s-c_{44}}{c_{13}+c_{44}} = \frac{(c_{13}+c_{44})s_\alpha}{c_{33}-c_{44}s}. \tag{7}$$

The remaining stress components are not cited in eqns (6), because we shall not need them in the subsequent analysis. The roots of eqn (5) may be either real (with the same sign) or complex conjugates. When $s_1$, $s_2$ are negative or complex conjugates, we choose $s_1^{1/2}$, $s_2^{1/2}$ to be complex conjugates with positive real parts.

## 3. Statement of the problem

Consider a transversely isotropic half-space reinforced by an elliptical disk. The disk is assumed to be inextensible and have no flexural stiffness at all. Complete bonding is assumed to exist between the half-space and the disk. Three different cases of loading of the disk have been considered: (a) The disk is subjected to the shearing force $T$, directed at an angle $\alpha$ to the major axis of the ellipse; (b) the disk is rotated at angle by a moment $M$, whose axis is perpendicular to the surface of the half-space (Fig. 1); (c) at infinity, the solid is subjected to tensile stress $p$ in a

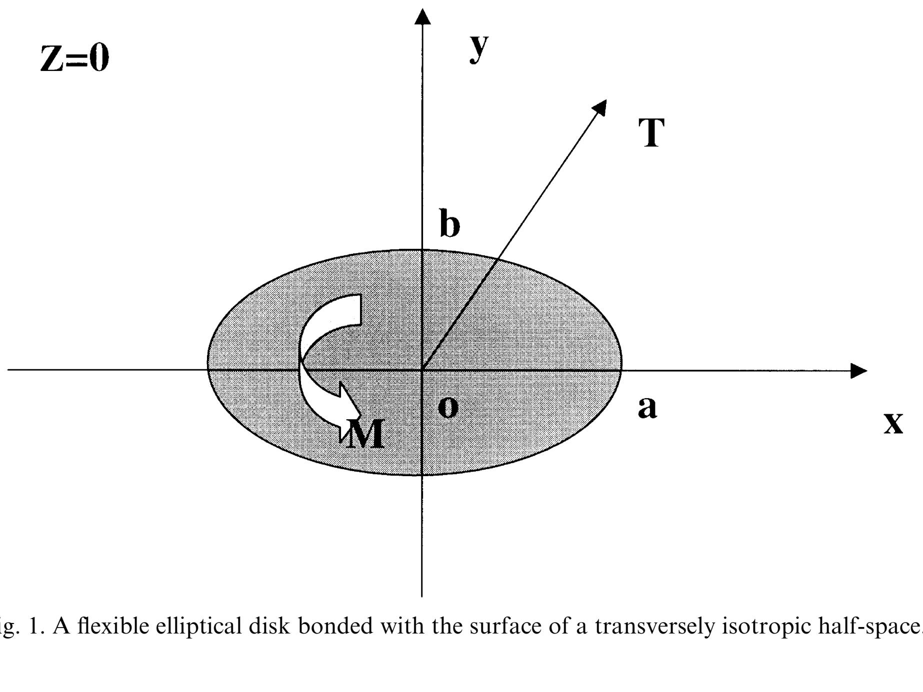


Fig. 1. A flexible elliptical disk bonded with the surface of a transversely isotropic half-space.

plane parallel to the plane of the disk, directed at angle $\gamma$ to the major axis of the ellipse. Let in the process of deformation of the system, the disk be rotated in its plane at an angle $\varphi$ and its center be displaced at a distance $\delta$ along a line whose direction makes an angle $\beta$ to the major axis of the ellipse. Thus, the problem consists in determining the contact stresses under the disk and also the relationships among the quantities $T$, $M$, $p$, $\alpha$, $\gamma$ and $\varphi$, $\delta$, $\beta$. We introduce the Cartesian coordinate system $x, y, z$ such that the region occupied by the half-space is given by the inequalities ($-\infty < x$, $y < \infty$, $0 \leqslant z < \infty$), and the contact region $\Omega$ between the disk and the elastic half-space by the inequality $1-x^2/a^2-y^2/b^2 \geqslant 0$ ($a \geqslant b$). We denote the compliment to the region $\Omega$ by $\tilde{\Omega}$. Within the framework of linear elasticity, we can split the problem into three smaller problems corresponding to the above three cases of loading. The boundary conditions for these problems are as follows:

Problem 1:

$$\begin{aligned} \sigma_{zz}(x,y,0) &= 0 \quad (x,y)\in\Omega\bigcup\tilde{\Omega}, \\ \sigma_{xz}(x,y,0) &= \sigma_{yz}(x,y,0) = 0, \quad (x,y)\in\tilde{\Omega}, \\ u(x,y,0) &= \delta\cos\beta, \quad (x,y)\in\Omega, \\ v(x,y,0) &= \delta\sin\beta, \quad (x,y)\in\Omega. \end{aligned} \tag{8}$$

Problem 2:

$$\begin{aligned} \sigma_{zz}(x,y,0) &= 0, \quad (x,y)\in\Omega\bigcup\tilde{\Omega}, \\ \sigma_{xz}(x,y,0) &= \sigma_{yz}(x,y,0) = 0, \quad (x,y)\in\tilde{\Omega}, \\ u(x,y,0) &= -\varphi y, \quad (x,y)\in\Omega, \\ v(x,y,0) &= \varphi x, \quad (x,y)\in\Omega. \end{aligned} \tag{9}$$

Problem 3:

$$\begin{aligned} \sigma_{zz}(x,y,0) &= 0, \quad (x,y)\in\Omega\bigcup\tilde{\Omega}, \\ \sigma_{xz}(x,y,0) &= \sigma_{yz}(x,y,0) = 0, \quad (x,y)\in\tilde{\Omega}, \\ u(x,y,0) &= -\varphi y, \quad (x,y)\in\Omega, \\ v(x,y,0) &= \varphi x, \quad (x,y)\in\Omega. \end{aligned} \tag{10}$$

At infinity, we have

$$\sigma_{xx} = p\cos^2\gamma, \quad \sigma_{yy} = p\sin^2\gamma, \quad \sigma_{xy} = \tfrac{1}{2}p\sin 2\gamma. \tag{11}$$

All other stress components vanish at infinity.

Within the framework of linear elasticity, the solution of Problem 3 can be obtained by superposing the solution of an unperturbed problem and a corrective solution. The unperturbed problem consists in finding the elastic field in the half-space without the disk under the boundary conditions (11), while the corrective problem consists in determining the elastic field in the medium in the presence of the disk subjected to the following boundary conditions:

$$
\begin{aligned}
\sigma_{zz}(x,y,0) &= 0, \quad (x,y)\in\Omega\bigcup\tilde{\Omega},\\
u(x,y,0) &= -\varphi y - u^0(x,y,0), \quad (x,y)\in\Omega,\\
v(x,y,0) &= \varphi x - v^0(x,y,0), \quad (x,y)\in\Omega,\\
\sigma_{xz}(x,y,0) &= \sigma_{yz}(x,y,0) = 0, \quad (x,y)\in\tilde{\Omega},
\end{aligned} \tag{12}
$$

where $u^0(x,y,0)$ and $v^0(x,y,0)$ are the solution of the unperturbed problem, the solution of which is given in Appendix A.

## 4. The solution

A suitable solution of the eqn (3) satisfying the regularity conditions is given by

$$
\chi_\alpha(x,y,z) = \mathscr{F}^{-1}[\{A_\alpha(\alpha_1,\alpha_2)\exp(-i\alpha_1 x - i\alpha_2 y - m_\alpha z);\alpha_1\to x\};\alpha_2\to y], \tag{13}
$$

where $A_\alpha$ ($\alpha = 1,2,3$) are some unknown constants to be determined using the boundary conditions of the problem and $m_\alpha = [(\alpha_1^2+\alpha_2^2)/s_\alpha]^{1/2}$.

Corresponding to (14), we have the following relations:

$$
\begin{aligned}
u(x,y,z) &= \mathscr{F}^{-1}[\{-i\alpha_1 A_1\exp(-m_1 z) - i\alpha_2 A_2\exp(-m_2 z) - i\alpha_2 A_3\\
&\qquad \exp(-m_3 z);\alpha_1\to x\};\alpha_2\to y],\\
v(x,y,z) &= \mathscr{F}^{-1}[\{-i\alpha_2 A_1\exp(-m_1 z) - i\alpha_2 A_2\exp(-m_2 z) + i\alpha_1 A_3\\
&\qquad \exp(-m_3 z);\alpha_1\to x\};\alpha_2\to y],\\
\sigma_{zz}(x,y,z) &= \mathscr{F}^{-1}[\{(k_1 c_{33} - s_1 c_{13})m_1^2 A_1\exp(-m_1 z)\\
&\qquad +(k_2 c_{33} - s_2 c_{13})m_2^2 A_2\exp(-m_2 z);\alpha_1\to z\};\alpha_2\to y],\\
\sigma_{xz}(x,y,z) &= \mathscr{F}^{-1}[\{ic_{44}(1+k_1)\alpha_1 A_1 m_1\exp(-m_1 z)\\
&\qquad + ic_{44}(1+k_2)\alpha_1 A_2 m_2\exp(-m_2 z) + ic_{44}\alpha_2 A_3 m_3\exp(-m_3 z);\alpha_1\to x\};\alpha_2\to y],\\
\sigma_{yz}(x,y,z) &= \mathscr{F}^{-1}[\{ic_{44}(1+k_1)\alpha_2 m_1 A_1\exp(-m_1 z) + ic_{44}(1+k_2)\\
&\qquad \alpha_2 m_2 A_2\exp(-m_2 z) - ic_{44}\alpha_1 A_3 m_3\exp(-m_3 z);\alpha_1\to x\};\alpha_2\to y].
\end{aligned} \tag{14}
$$

Now using the stress boundary conditions of the problem [see eqns (8)–(10)] and the eqns (14), it can be shown that

$$
\begin{aligned}
u(x,y,0) = \mathscr{F}^{-1}[\{&-c_{44}^{-1}\Lambda\tilde{\sigma}_{xz}(\alpha_1,\alpha_2,0)(\alpha_1^2+\alpha_2^2)^{1/2} - c_{44}^{-1}(s_3^{1/2}-\Lambda)\times\\
&\alpha_2^2(\alpha_1^2+\alpha_2^2)^{-(3/2)}\tilde{\sigma}_{xz}(\alpha_1,\alpha_2,0) + c_{44}^{-1}(s_3^{1/2}-\Lambda)\alpha_1\alpha_2(\alpha_1^2+\alpha_2^2)^{-(3/2)}\tilde{\sigma}_{yz}(\alpha_1,\alpha_2,0);\\
&\alpha_1\to x\};\alpha_2\to y],
\end{aligned}
$$

$$v(x,y,0)=\mathscr{F}^{-1}[\{-c_{44}^{-1}\Lambda\tilde{\sigma}_{yz}(\alpha_1,\alpha_2,0)(\alpha_1^2+\alpha_2^2)^{-(1/2)}-c_{44}^{-1}(s_3^{1/2}-\Lambda)$$
$$\alpha_2^2(\alpha_1^2+\alpha_2^2)^{-(3/2)}\tilde{\sigma}_{yz}(\alpha_1,\alpha_2,0)+c_{44}^{-1}(s_3^{1/2}-\Lambda)\alpha_1\alpha_2(\alpha_1^2+\alpha_2^2)^{-(3/2)}\tilde{\sigma}_{xz}(\alpha_1,\alpha_2,0);$$
$$\alpha_1\to x\};\alpha_2\to y], \tag{15}$$

where

$$\Lambda=\frac{1-\Gamma}{s_1^{-(1/2)}(1+k_1)-s_2^{-(1/2)}(1+k_2)\Gamma},$$
$$\Gamma=\frac{s_2(k_1c_{33}-s_1c_{13})}{s_1(k_2c_{33}-s_2c_{13})}. \tag{16}$$

Using convolution theorem for Fourier transform, we obtain

$$u(x,y,0)=-\frac{\Lambda}{2\pi c_{44}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\sigma_{xz}(\xi,\eta)}{R}\,\mathrm{d}\xi\,\mathrm{d}\eta-\frac{s_3^{1/2}-\Lambda}{2\pi c_{44}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\sigma_{xz}(\xi,\eta)}{R^3}(x-\xi)^2\,\mathrm{d}\xi\,\mathrm{d}\eta$$
$$-\frac{s_3^{1/2}-\Lambda}{2\pi c_{44}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\sigma_{yz}(\xi,\eta)}{R^3}(x-\xi)(y-\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta,$$
$$v(x,y,0)=-\frac{\Lambda}{2\pi c_{44}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\sigma_{yz}(\xi,\eta)}{R}\,\mathrm{d}\xi\,\mathrm{d}\eta-\frac{s_3^{1/2}-\Lambda}{2\pi c_{44}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\sigma_{xz}(\xi,\eta)}{R^3}(y-\eta)^2\,\mathrm{d}\xi\,\mathrm{d}\eta$$
$$-\frac{s_3^{1/2}-\Lambda}{2\pi c_{44}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\sigma_{xz}(\xi,\eta)}{R^3}(x-\xi)(y-\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta, \tag{17}$$

where $R=\{(x-\xi)^2+(y-\eta)^2\}^{1/2}$.

Equations (17) give the surface displacements of a transversely isotropic half-space loaded on the surface by the shear stresses $\sigma_{xx}(x,y,0)$ and $\sigma_{yz}(x,y,0)$.

In deriving eqns (18), use has been made of the following integrals (Erdelyi, 1954):

$$\mathscr{F}^{-1}[\{(\alpha_1^2+\alpha_2^2)^{1/2};\alpha_1\to x\};\alpha_2\to y]=\frac{1}{R},$$
$$\mathscr{F}^{-1}[\{\alpha_1^2/(\alpha_1^2+\alpha_2^2)^{1/2};\alpha_1\to x\};\alpha_2\to y]=\frac{y^2}{R^3},$$
$$\mathscr{F}^{-1}[\{\alpha_2^2/(\alpha_1^2+\alpha_2^2)^{1/2};\alpha_1\to x\};\alpha_2\to y]=\frac{x^2}{R^3},$$
$$\mathscr{F}^{-1}[\{\alpha_1\alpha_2/(\alpha_1^2+\alpha_2^2)^{1/2};\alpha_1\to x\};\alpha_2\to y]=\frac{-xy}{R^3}.$$

Now invoking the displacement boundary conditions of the problem [see eqns (8)–(10)], we obtain the following integral equations to determine the contact stresses $\sigma_{xz}(x,y,0)$ and $\sigma_{yz}(x,y,0)$ under the disk:

$$\iint\limits_{\Omega}\left[\frac{1}{R}+\kappa\frac{(x-\xi)^2}{R^3}\right]\sigma_{xz}(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta+\kappa\iint\limits_{\Omega}\frac{(x-\xi)(y-\eta)}{R^3}\sigma_{yz}(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta$$

$$=\pi(a_0+a_1x+a_2y),\quad (x,y)\in\Omega,$$

$$\iint\limits_{\Omega}\left[\frac{1}{R}+\kappa\frac{(y-\eta)^2}{R^3}\right]\sigma_{yz}(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta+\kappa\iint\limits_{\Omega}\frac{(x-\xi)(y-\eta)}{R^3}\sigma_{xz}(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta$$

$$=\pi(b_0+b_1x+b_2y),\quad (x,y)\in\Omega,\quad \kappa=\frac{s_3^{1/2}-\Lambda}{\Lambda} \tag{18}$$

where the constants $a_i$ and $b_i$ ($i=0,1,2$) are given by

Problem 1:

$$a_0=-\frac{2c_{44}\,\delta\cos\beta}{\Lambda},\quad b_0=-\frac{2c_{44}\,\delta\sin\beta}{\Lambda},\quad a_1=b_1=a_2=b_2=0, \tag{19}$$

Problem 2:

$$a_2=\frac{2c_{44}\varphi}{\Lambda},\quad b_1=\frac{-2c_{44}\varphi}{\Lambda},\quad a_0=a_1=b_0=b_2=0, \tag{20}$$

Problem 3:

$$a_0=b_0=0,$$

$$a_1=-\frac{2c_{44}p(\Lambda_1\cos^2\gamma-\Lambda_2\sin^2\gamma)}{\Lambda},$$

$$a_2=\frac{2c_{44}\varphi}{\Lambda}-\frac{2c_{44}p\Lambda_3\sin 2\gamma}{\Lambda},$$

$$b_1=-\frac{2c_{44}\varphi}{\Lambda}-\frac{2c_{44}p\Lambda_3\sin 2\gamma}{\Lambda},$$

$$b_2=-\frac{2c_{44}p(\Lambda_1\sin^2\gamma-\Lambda_2\cos^2\gamma)}{\Lambda}. \tag{21}$$

Following Alexandrov and Solov'ev (1966), we introduce the notations

$$W_{ij}=\iint\limits_{\Omega}\frac{\xi_i\eta_j}{Rl(\xi,\eta)}\mathrm{d}\xi\,\mathrm{d}\eta,\quad A_{ij}=\iint\limits_{\Omega}\frac{(x-\xi)^2\xi^i\eta^j}{R^3l(\xi,\eta)}\mathrm{d}\xi\,\mathrm{d}\eta,\quad l(\xi,\eta)=\left(1-\frac{\xi^2}{a^2}-\frac{\eta^2}{b^2}\right)^{1/2},$$

$$B_{ij}=\iint\limits_{\Omega}\frac{(y-\eta)^2\xi^i\eta^j}{R^3l(\xi,\eta)}\mathrm{d}\xi,\mathrm{d}\eta,\quad C_{ij}=\iint\limits_{\Omega}\frac{(x-\xi)(y-\eta)\xi^i\eta^j}{R^3l(\xi,\eta)}\mathrm{d}\xi\,\mathrm{d}\eta. \tag{22}$$

It can be verified by direct computation that the following relations hold:

$$A_{ij} = (W_{i+1,j})'_x - x(W_{ij})'_x,$$
$$B_{ij} = (W_{i,j+1})'_y - y(W_{ij})'_y,$$
$$C_{ij} = (W_{i+1,j})'_y - x(W_{ij})'_x = (W_{i,j+1})'_x - y(W_{ij})'_x. \quad (23)$$

where the primes denote differentiation with respect to the coordinate shown in the subscripts. It is obvious from eqns (23) that we need to evaluate the integral $W_{ij}$ only. All other integrals, viz. $A_{ij}$, $B_{ij}$, $C_{ij}$, can be evaluated using the relations (23). The evaluation of the integral $W_{ij}$ has been discussed at greater length in the book (Vorovich et al., 1974). The readers are referred to Appendix B where a synopsis of these results has been given.

To solve the integral eqns (18), we use Galin's theorem (Galin, 1976; Gladwell, 1980) (see also Appendix C) and represent the unknown contact pressures in the form:

$$\sigma_{xz}(x, y, 0) = l^{-1}(x, y)(c_0 + c_1 x + c_2 y),$$
$$\sigma_{yz}(x, y, 0) = l^{-1}(x, y)(d_0 + d_1 x + d_2 y). \quad (24)$$

Putting (24) into (18), we obtain the following algebraic equations to determine the unknown coefficients $c_i$, $d_i$:

$$(W_{00} + \kappa A_{00})c_0 + (W_{10} + \kappa A_{10})c_1 + (W_{01} + \kappa A_{01})c_2 + \kappa C_{00} d_0$$
$$+ \kappa C_{01} d_1 + \kappa C_{01} d_2 = \pi(a_0 + a_1 x + a_2 y),$$
$$(W_{00} + \kappa B_{00})d_0 + (W_{10} + \kappa B_{10})d_1 + (W_{01} + \kappa B_{01})d_2 + \kappa C_{00} c_0$$
$$+ \kappa C_{10} c_1 + \kappa C_{01} c_2 = \pi(b_0 + b_1 x + b_2 y). \quad (25)$$

It can be shown by using the equations (23) and the closed-form expression for the integral $W_{ij}$ (see Appendix B) that the following relations hold:

$$W_{00} + \kappa A_{00} = \pi b P_{00}, \quad W_{10} + \kappa A_{10} = \pi b x P_{10}, \quad W_{01} + \kappa A_{01} = \pi b y P_{01},$$
$$W_{00} + \kappa B_{00} = \pi b Q_{00}, \quad W_{10} + \kappa B_{10} = \pi b x Q_{10}, \quad W_{01} + \kappa B_{01} = \pi b y Q_{01},$$
$$C_{00} = 0, \quad C_{10} = \pi b y R, \quad C_{01} = \pi b x S, \quad (26)$$

where

$$P_{00} = \frac{2}{e^2}[(e^2 + \kappa)K(e) - \kappa E(e)],$$

$$Q_{00} = \frac{2}{e^2}[(e^2 + \kappa e^2 - \kappa)K(e) + \kappa E(e)],$$

$$P_{10} = \frac{2}{e^4}[e^2 - \kappa(e^2 + 2)K(e) - (2\kappa + e^2)E(e)],$$

$$P_{01} = \frac{2}{e^4}[-(2\kappa + e^2)(1 - e^2)K(e) + (e^2 - \kappa e^2 + 2\kappa)E(e)],$$

$$Q_{10} = \frac{2}{e^4}[(e^2+2\kappa e^2-2\kappa)K(e)-(e^2+\kappa e^2-2\kappa)E(e)],$$

$$Q_{01} = \frac{2}{e^4}[-(e^2+\kappa e^2-2\kappa)(1-e^2)K(e)+(e^2+2\kappa e^2-2\kappa)E(e)],$$

$$R = -\frac{2}{e^4}[(2-e^2)K(e)-2E(e)], \quad S = -\frac{2(1-e^2)}{e^4}[(2-e^2)K(e)-2E(e)],$$

where $e$ is the eccentricity of the ellipse and $K(e)$, $E$(e) are the complete elliptical integrals of the first and second kinds, respectively.

In passing, we note that for a circular domain (i.e. $e = 0$), eqns (27) reduce to

$$P_{00} = Q_{00} = \frac{\pi}{2}(2+\kappa), \quad P_{10} = Q_{01} = \frac{\pi}{8}(4+\kappa),$$

$$P_{01} = Q_{10} = \frac{\pi}{8}(4+3\kappa), \quad R = S = -\frac{\pi}{8}. \tag{28}$$

Equations (28) can be derived by letting $e \to 0$ in (27) and using L'Hopital's rule in conjunction with the rule for differentiation of the complete elliptical integrals of the first and second kinds.

Substituting the relations (27) into (25) and equating the terms of the left and right hand sides with like powers of $x$ and $y$ yields a system of linear algebraic equations, solving which we get the following expressions for the unknown coefficients $c_i$, $d_i$:

$$c_0 = \frac{a_0}{bP_{00}}, \quad c_1 = \frac{a_1 Q_{01} - b_2\kappa S}{b(P_{10}Q_{01}-\kappa^2 RS)}, \quad c_2 = \frac{a_2 Q_{10} - b_1 \kappa R}{b(P_{01}Q_{10}-\kappa^2 RS)},$$

$$d_0 = \frac{b_0}{bQ_{00}}, \quad d_1 = \frac{b_1 P_{01} - a_2\kappa S}{b(P_{01}Q_{10}-\kappa^2 RS)}, \quad d_2 = \frac{b_2 P_{10} - a_1\kappa R}{b(P_{10}Q_{01}-\kappa^2 RS)}. \tag{29}$$

We now proceed to the detailed consideration of all three problems.

Problem 1: For this case, the constants $a_i$, $b_i$ are given by (19). Then using eqns (24) and (29), we obtain the following expressions for the contact stresses:

$$\sigma_{xz}(x,y,0) = \frac{-2c_{44}\,\delta\cos\beta}{\Lambda bP_{00}l(x,y)}, \quad \sigma_{yz}(x,y,0) = \frac{-2c_{44}\,\delta\sin\beta}{\Lambda bQ_{00}l(x,y)}. \tag{30}$$

The components of the shearing force $T$ along the $x$- and $y$-axes are (see Appendix B):

$$T_x = \iint_\Omega \sigma_{xz}(x,y,0)\,\mathrm{d}x\,\mathrm{d}y = \frac{-4c_{44}\,\delta a\pi\cos\beta}{\Lambda P_{00}},$$

$$T_y = \iint_\Omega \sigma_{yz}(x,y,0)\,\mathrm{d}x\,\mathrm{d}t = \frac{-4c_{44}\delta a\pi\sin\beta}{\Lambda Q_{00}}. \tag{31}$$

Therefore, the resultant transverse force is

$$T = (T_x^2 + T_y^2)^{1/2} = \frac{4c_{44}\delta a\pi}{\Lambda}\left(\frac{\cos^2\beta}{P_{00}^2} + \frac{\sin^2\beta}{Q_0^2}\right)^{1/2}. \tag{32}$$

The slope of the force $T$ relative to the $x$-axis is

$$\tan\alpha = \frac{T_y}{T_x} = \frac{P_{00}}{Q_{00}}\tan\beta. \tag{33}$$

Next, consider some special cases of this problem.

Case A: Let $\alpha = 0$, that is, the line of action of the transverse force coincides with the $x$-axis. Then $T = T_x$, $\beta = 0$, $T_y = 0$. Therefore, the equations for the contact stresses (30) reduce to the following:

$$\sigma_{xz}(x,y,0) = \frac{-2c_{44}\,\delta a}{\Lambda P_{00}l(x,y)}, \quad \sigma_{yz}(x,y,0) = 0, \quad T = \frac{4c_{44}\,\delta a\pi}{\Lambda P_{00}}. \tag{34}$$

Case B: Let $\alpha = \pi/2$, which means that the line of action of the transverse force is along the $y$-axis. Then $T = T_y$, $\beta = (\pi/2)$, $T_x = 0$. Therefore, the contact stresses are given by

$$\sigma_{yz}(x,y,0) = \frac{-2c_{44}\,\delta a}{\Lambda Q_{00}l(x,y)}, \quad \sigma_{xz}(x,y,0) = 0, \quad T = \frac{4c_{44}\,\delta a\pi}{\Lambda Q_{00}}. \tag{35}$$

Case C: Consider the case of a circular disk, i.e. $a = b$ $(e = 0)$. Then $\alpha = \beta$ and we have the following equations

$$\frac{\sigma_{xz}(x,y,0)}{\cos\beta} = \frac{\sigma_{yz}(x,y,0)}{\sin\beta} = \frac{-2c_{44}\delta}{\Lambda bQ_{00}l(x,y)} = \frac{-2c_{44}\delta}{\pi^2(2+\kappa)b\Lambda l(x,y)},$$

$$T = \frac{4c_{44}\delta a\pi}{\Lambda}\left(\frac{\cos^2\beta}{P_{00}^2} + \frac{\sin^2\beta}{Q_{00}^2}\right)^{1/2} = \frac{4c_{44}\delta a}{\pi(2+\kappa)\Lambda}. \tag{36}$$

In deriving equations (36), use has been made of the relations (28).

We now proceed to consider the second problem.

Problem 2: In this case, the constants $a_i$, $b_i$ are given by (20), which upon substituting into equations (24) and (29) yields the following equations for determining the contact stresses:

$$\sigma_{xz}(x,y,0) = \frac{2c_{44}\varphi(Q_{10}+\kappa R)y}{b\Lambda(P_{01}Q_{10}-\kappa^2 RS)l(x,y)},$$

$$\sigma_{yz}(x,y,0) = \frac{-2c_{44}\varphi(P_{01}+\kappa S)x}{b\Lambda(P_{01}Q_{10}-\kappa^2 RS)l(x,y)}. \tag{37}$$

The moment $M$ and the angle of rotation $\varphi$ are related by the equation

$$M = \iint_{\Omega} [x\sigma_{yz}(x,y,0) - y\sigma_{xz}(x,y,0)]\,\mathrm{d}x\,\mathrm{d}y$$

$$= \frac{-4c_{44}a\varphi\pi}{3\Lambda(P_{01}Q_{10}-\kappa^2 RS)}[(P_{01}+\kappa S)a^2 + (Q_{10}+\kappa R)b^2]. \tag{38}$$

With regard to the evaluation of the integral (38), the readers are referred to Appendix D.

The solution corresponding to the case of a circular disk can be obtained by letting $e \to 0$ in eqns (37) and (38) and using the results (28), viz

$$\sigma_{0z}(r) = \{\sigma_{xz}^2(x,y,0) + \sigma_{yz}^2(x,y,0)\}^{1/2} = \frac{4c_{44}\varphi r}{\pi a\Lambda(1+\kappa)}\left(1-\frac{r^2}{a^2}\right)^{-1/2},$$

$$M = \frac{-16c_{44}\varphi a^3}{3\Lambda(1+\kappa)}. \tag{39}$$

Problem 3: For this problem, the constants $a_i$, $b_i$ are given by eqns (21). Substituting into eqns (24) and (29), we obtain the following equations to determine the contact stresses:

$$\begin{aligned}
\sigma_{xz}(x,y,0) = &-\frac{2px}{\Lambda b(P_{10}Q_{01}-\kappa^2 RS)}[Q_{01}(\Lambda_1\cos^2\gamma - \Lambda_2\sin^2\gamma) \\
&-\kappa S(\Lambda_1\sin^2\gamma - \Lambda_2\cos^2\gamma)]l^{-1}(x,y) + \frac{2y}{\Lambda b(P_{01}Q_{10}-\kappa^2 RS)} \\
&\times[(c_{44}\varphi - \Lambda_3 p\sin 2\gamma)Q_{10} + \kappa(c_{44}\varphi + \Lambda_3 p\sin 2\gamma)R]l^{-1}(x,y), \\
\sigma_{yz}(x,y,0) = &-\frac{2x}{\Lambda b(P_{01}Q_{10}-\kappa^2 RS)}[P_{01}(c_{44}\varphi + \Lambda_3 p\sin 2\gamma) \\
&+\kappa S(c_{44}\varphi - \Lambda_3 p\sin 2\gamma)]l^{-1}(x,y) - \frac{2py}{\Lambda b(P_{10}Q_{01}-\kappa^2 RS)} \\
&\times[P_{10}(\Lambda_1\sin^2\gamma - \Lambda_2\cos^2\gamma) - \kappa R(\Lambda_1\cos^2\varphi - \Lambda_2\sin^2\gamma)]l^{-1}(x,y).
\end{aligned} \tag{40}$$

We find the angle of rotation $\varphi$ of the disk from the condition that the surface is free of load, i.e. $T_x = T_y = 0$, $M = 0$.

The second condition $M = 0$ gives the required relationship between $p$, $\alpha$ and $\varphi$, namely,

$$\varphi = \frac{-\Lambda_3\sin 2\gamma[(P_{01}-S)a^2 - (Q_{10}-R)b^2]}{(P_{01}+\kappa S)a^2 + (Q_{10}+\kappa R)b^2}. \tag{41}$$

Equations (40) and (41) complete the solution of Problem 3.

Next we consider some special cases of this problem. The case where $\gamma = 0$ corresponds to the action of tensile stresses $p$ on the elastic half-space in a direction parallel to the $x$-axis. Accordingly, for this case $\varphi = 0$ and the equations for the contact stresses (40) reduce to

$$\sigma_{xz}(x,y,0) = \frac{-2px(\Lambda_1 Q_{01}+\Lambda_2 S)}{\Lambda b(P_{01}Q_{10}-\kappa^2 RS)l(x,y)},$$
$$\sigma_{yz}(x,y,0) = \frac{-2py(\Lambda_2 P_{10}+\Lambda_1 R)}{\Lambda b(P_{10}Q_{01}-\kappa^2 RS)l(x,y)}. \tag{42}$$

The case where $\gamma = \pi/2$ corresponds to the action of tensile stresses on the elastic half-space in a direction parallel to the $y$-axis. Also, in this case we have $\varphi = 0$ and the contact stresses are given by the equations

$$\sigma_{xz}(x,y,0) = \frac{2px(\Lambda_2 Q_{01}+\Lambda_1 S)}{\Lambda b(P_{01}Q_{10}-\kappa^2 RS)l(x,y)},$$
$$\sigma_{yz}(x,y,0) = \frac{-2py(\Lambda_1 P_{10}+\Lambda_2 R)}{\Lambda b(P_{10}Q_{01}-\kappa^2 RS)l(x,y)}. \tag{43}$$

Finally, let us consider the case of a circular disk as a limiting case of the solution (40). In this case, it follows that $\varphi = 0$ and the equations determining the contact stresses assume the following form:

$$\sigma_{xz}(x,y,0) = \frac{-2px}{\pi a\xi(2+\kappa)}[\{\Lambda_1(4+\kappa)-\Lambda_2\kappa\}\cos^2\gamma+\{\Lambda_2(4+\kappa)-\Lambda_1\kappa\}\sin^2\gamma]$$
$$\times\left(1-\frac{r^2}{a^2}\right)^{-1/2} - \frac{8py\Lambda_3}{\pi a\Lambda(2+\kappa)}\sin 2\gamma\left(1-\frac{r^2}{a^2}\right)^{-1/2},$$
$$\sigma_{yz}(x,y,0) = \frac{-2py}{\pi a\Lambda(2+\kappa)}[\{\Lambda_1(4+\kappa)-\Lambda_2\kappa\}\sin^2\gamma+\{\Lambda_2(4+\kappa)-\Lambda_1\kappa\}\cos^2\gamma]$$
$$\times\left(1-\frac{r^2}{a^2}\right)^{-1/2} - \frac{8px\Lambda_3}{\pi a\Lambda(2+\kappa)}\sin 2\gamma\left(1-\frac{r^2}{a^2}\right)^{-1/2}. \tag{44}$$

In deriving eqns (44), use has been made of the relations (28).

## 5. The special case of isotropy

For an isotropic material, we have (Green and Zerna, 1968)

$$c_{11} = c_{33} = \frac{2(1-v)\mu}{1-2v}, \quad c_{12} = c_{13} = \frac{2v\mu}{1-2v}, \quad c_{44} = \mu, \tag{45}$$

where $\mu$ and $v$ are the shear modulus and Poisson's ratio. With (45), we get, using eqns (4), (5) and (A.3),

$$s_1 = s_2 = s_3 = 1,$$

$$\Lambda_1 = \frac{1}{2(1+\nu)}, \quad \Lambda_2 = \frac{\nu}{2(1+\nu)}, \quad \Lambda_3 = \frac{1}{4}, \quad \Lambda_4 = \frac{\nu}{2(1+\nu)}. \tag{46}$$

If we next put $s_1 = s_2 = s_3 = 1$ into the equations for $\Gamma$ and $\Lambda$ [eqn (16)] and $\kappa$ [eqn (18)], we see that they reduce to indeterminacies. In order to overcome this difficulty, we assume that $s_{1,2} = 1 \pm i\varepsilon$, where $\varepsilon$ is a small positive quantity. With this, we have the following expressions

$$k_{1,2} = 1 \pm 2i\varepsilon(1-\nu), \quad \Gamma = 1+2i\varepsilon(1-\nu), \quad \Lambda = 1-\nu, \quad \kappa = \frac{\nu}{1-\nu}. \tag{47}$$

Putting (46) and (47) into eqns (30)–(44) and passing to the limit $\varepsilon \to 0$, we find the solution corresponding to the isotropic case (which we do not list here for the sake of saving space) and observe that they are precisely the same as those obtained by Alexandrov and Solov'ev (1966) except a minor sign error in their equation (38).

## 6. Closure

In the present paper, we have considered the contact problem of a flexible elliptical disk bonded to the surface of a transversely isotropic half-space under three different cases of loading, namely, (a) the disk is loaded by a transverse load in its plane; (b) the disk is subjected to a rotation by a concentrated moment; (d) the disk-half-space system is under uniform stress field at infinity acting in a plane parallel to that of the disk. The problem corresponding to all three cases has been reduced, in a unified manner, to a set of coupled two-dimensional integral equations, the exact solution of which have been found by using Galin's theorem. To the best of the author's knowledge, the present solution is new. The correctness of the solution has been verified by comparing them to those known for the isotropic case. Of further interest is the analogous problem under arbitrary polynomial loading, for which the present approach based on Galin's theorem is most suitable. Another issue of considerable interest is the corresponding problem involving buried loads in the transversely isotropic half-space, in which case the right hand sides of the integral equations will no longer be polynomial. But, we note that since any continuous function in a bounded region (which in our case is elliptical) can be approximated up to any accuracy by polynomials of $x$ and $y$ (Bernstein's theorem), Galin's theorem can still be applied to solve the problem. Research in this direction will be reported elsewhere.

## Appendix A

The objective of this Appendix is to derive the solution for the unperturbed problem corresponding to the third case of loading.

The solution of the problem is sought in the form:

$$u(x,y,z) = p_1x+t_1y+r_1z$$
$$v(x,y,z) = p_2x+t_2y+r_2z$$
$$w(x,y,z) = p_3x+t_3y+r_3z \tag{A1}$$

where $p_i$, $t_i$, $r_i$ $(i = 1, 2, 3)$ are some constants. We note that in view of the linearity of these expressions, the equilibrium eqns (1) are automatically satisfied. Next, invoking the boundary conditions (11) and the stress-free conditions on the surface of the half-space, viz, $\sigma_{zz}(x, y, 0) = \sigma_{xz}(x, y, 0) = \sigma_{yz}(x, y, 0) = 0$, we obtain the following solution:

$$u(x,y,z) = c_{44}^{-1}p(\Lambda_1 \cos^2\gamma - \Lambda_2 \sin^2\gamma)x + c_{44}^{-1}p\Lambda_3 \sin 2\gamma\, y,$$

$$v(x,y,z) = c_{44}^{-1}p\Lambda_3 x \sin 2\gamma + c_{44}^{-1}p(\Lambda_1 \sin^2\gamma - \Lambda_2 \cos^2\gamma)\gamma,$$

$$w(x,y,z) = -c_{44}^{-1}p\Lambda_4 z, \tag{A2}$$

where

$$\Lambda_1 = \frac{c_{44}(c_{11}c_{33} - c_{13}^2)}{(c_{11} - c_{12})(c_{11}c_{33} + c_{12}c_{33} - 2c_{13}^2)},$$

$$\Lambda_2 = \frac{c_{44}(c_{12}c_{33} - c_{13}^2)}{(c_{11} - c_{12})(c_{11}c_{33} + c_{12}c_{33} - 2c_{13}^2)},$$

$$\Lambda_3 = \frac{c_{44}}{2(c_{11} - c_{12})},$$

$$\Lambda_4 = \frac{c_{13}c_{44}}{c_{11}c_{33} + c_{12}c_{13} - 2c_{13}^2}. \tag{A3}$$

## Appendix B

In this Appendix, we give a synopsis of the results concerning the evaluation of the integral $W_{ij}$, namely,

$$W_{ij} = \iint_{\Omega} \frac{\xi^i \eta^j}{\{(x-\xi)^2 + (y-\eta)^2\}^{1/2}} \left(1 - \frac{\xi^2}{a^2} - \frac{\eta^2}{b^2}\right)^{-1/2} \mathrm{d}\xi\, \mathrm{d}\eta, \tag{B1}$$

where $\Omega$ is the elliptical domain $(1 - x^2/a^2 - y^2/b^2)^{1/2} \leqslant 0$. Evaluation of this integral has been discussed at great length by Vorovich et al. (1976), so without going into details, we give the final results below:

$$W_{ij} = \int_0^{\pi} \mathrm{d}\varphi \int_0^{\pi} \left(x + \frac{K\cos\varphi\cos\theta}{L^{1/2}} \frac{M\cos\varphi}{L}\right)^i \left(y + \frac{K\sin\varphi\cos\theta}{L^{1/2}} - \frac{M\sin\varphi}{L}\right)^j \frac{\mathrm{d}\theta}{L^{1/2}}, \tag{B2}$$

where the following notations are introduced:

$$L(\varphi) = \frac{\cos^2\varphi}{a^2} + \frac{\sin^2\varphi}{b^2}, \quad M(\varphi) = \frac{x\cos\varphi}{a^2} + \frac{y\sin\varphi}{b^2}, \quad N = 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}. \tag{B3}$$

Integrating with respect to $\theta$ and changing the variable $\varphi = \varphi' + \pi/2$, after some algebraic manipulations, we reduce equation to the final form:

$$W_{ij} = \sum_{r=0}^{i}\sum_{s=0}^{j} C_i^r C_j^s B\left(\frac{r+s+1}{2},\frac{1}{2}\right)(1-e^2)^{\frac{r-s+2j+1}{2}} \sum_{q=0}^{(r+s)/2} C_{(r+s)/2}^{q}(-1)^{(r-s)/2} a^{2q+1}$$

$$\times \sum_{p-0}^{i+j-2q} C_{i+j-2q}^{p} x^{i+j-2q-p} y^p S_{i-q+\frac{j-p+s-r}{2},\frac{j+p-s+r}{2}}, \quad \text{(B4)}$$

where $r+s$ and $j+p$ are even, $C_n^k = n!/\{k!(n-k)!\}$, $B(x, y)$ is the Beta function, and

$$S_{m,n} = \int_0^{\pi} \frac{\cos^{2m}\varphi \sin^{2n}\varphi}{(1-e^2\sin^2\varphi)^{m+n+(1/2)}} \mathrm{d}\varphi. \quad \text{(B5)}$$

Integrals $S_{m,n}$ for $\forall m, n \geqslant 0$ can be expressed in terms of the complete elliptical integrals of the first and second kinds by means of the recurrence relations derived by Mayrhoffer and Fischer (1994).

## Appendix C

Consider the integral equation

$$\iint_{\Omega} \frac{p(x_0, y_0)}{R} \mathrm{d}x_0\, \mathrm{d}y_0 = \pi f(x, y), \quad \text{(C1)}$$

where $R = [(x-x_0)^2+(y-y_0)^2]^{1/2}$ and $\Omega$ is an elliptical region. Galin's theorem states that if the function $f(x, y)$ can be represented as

$$f(x, y) = \sum_{m=0}^{p}\sum_{n=0}^{q} a_{mn} x^m y^n, \quad \text{(C2)}$$

then the solution of the integral equation has the following form

$$p(x, y) = \left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^{-1/2} \sum_{m=0}^{p}\sum_{n=0}^{q} b_{mn} x^m y^n, \quad \text{(C3)}$$

where $a$ and $b$ are the major and minor semi-axes of the ellipse. We now proceed to show that although the integral eqns (18) are not in the form of (C1), nonetheless the same results hold for them too. Indeed, consider the following integral equation

$$\iint_{\Omega} \frac{p(x_0, y_0)}{R^3}(x-x_0)^2\, \mathrm{d}x_0\, \mathrm{d}y_0 = \pi f(x, y). \quad \text{(C4)}$$

Equation (C4) can be written as

$$-x\frac{\partial}{\partial x}\iint_{\Omega} \frac{p(x_0, y_0)}{R} \mathrm{d}x_0\, \mathrm{d}y_0 + \frac{\partial}{\partial x}\iint_{\Omega} \frac{x_0 p(x_0, y_0)}{R} \mathrm{d}x_0\, \mathrm{d}y_0 = \pi f(x, y). \quad \text{(C5)}$$

Now, it is obvious that the results (C2) and (C3) do also hold for eqn (C4). In a similar fashion, it can be shown the same results also hold for the integral equations

$$\iint_{\Omega} \frac{p(x_0, y_0)}{R^3}(y-y_0)^2 \, dx_0 \, dy_0 = \pi f(x, y), \tag{C6}$$

$$\iint_{\Omega} \frac{p(x_0, y_0)}{R^3}(x-x_0)(y-y_0) \, dx_0 \, dy_0 = \pi f(x, y), \tag{C7}$$

thus proving that results of the type (C2) and (C3) hold also for the integral eqns (18).

## Appendix D

In deriving eqns (31) and (38), use has been made of the integral

$$\iint_{\Omega} x^{2m} y^{2n} \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{-1/2} dx \, dy = 2\pi a^{2m+1} b^{2n+1} \frac{(2m-1)!!(2n-1)!!}{(2m+2n+1)!!}, \tag{D1}$$

where it is assumed that $(-1)!! = 1$. Derivation of the integral (D1) can be found in many books on contact mechanics, e.g. Galin (1976), Gladwell (1980).

## References


Alexandrov, V.M., Solov'ev, A.S., 1966. Some mixed three-dimensional problems in the theory of elasticity. Mechanics of Solids (Mekhanika Tverdogo Tela) 2, 95–97.

Borodachev, N.M., 1990. Intrusion of die with a near-circular base in a transversely isotropic half-space. Soviet Applied Mechanics 26 (7), 642–647.

Conway, H.D., Farnham, K.A., Ku, T.C., 1967a. The indentation of a transversely isotropic half-space by a rigid sphere. ASME Journal of Applied Mechanics 34, 491–492.

Conway, H.D., Farnham, K.A., 1967b. Shearing stresses induced on an axis of symmetry by circular sliding contact on a transversely isotropic body. ASME Journal of Applied Mechanics 34, 756–757.

Dahan, M., Zarka, J., 1977. Elastic contact between a sphere and semi-infinite transversely isotropic body. International Journal of Solids and Structures 13, 229–238.

Elliot, H.A., 1948. Three-dimensional stress distribution in hexagonal aelotropic crystals. Proceedings of the Cambridge Philosophical Society 44, 522–533.

Elliot, H.A., 1949. Axial symmetric stress distributions in aelotropic hexagonal crystals. The problem of the plane and related problems. Proceedings of the Cambridge Philosophical Society 45, 621–630.

Erdelyi, A., 1954. Tables of Integral Transforms, vol. I. McGraw-Hill, New York.

Fabrikant, V.I., 1997. Exact solution of tangential contact problem for a circular domain. Journal of the Mechanics and Physics of Solids 45 (1) 113–134.

Galin, L.A. (Ed.), 1976. Development of the Theory of Contact Problems in the USSR. Nauka, Moscow (in Russian).

Gladwell, G.M.L., 1980. Contact Problems in the Classical Theory of Elasticity. Sijhoff and Noordhoff, Alphen aan den Rijn, The Netherlands.

Gradshteyn, I.M., Ryzhik, I.S., 1994. Tables of Integrals, Series and Products. Academic Press, New York.

Green, A.E., Zerna, W., 1968. Theoretical Elasticity. Oxford University Press.

Guidera, J.T., Lardner, R.W., Tupholme, G.E., 1978. The indentation of a half-space of hexagonal elastic material by a circular punch of arbitrary end-profile. Journal of Engineering Mathematics 12, 77–82.

Lure, A.I., 1964. Three-dimensional Problems of the Theory of Elasticity. Interscience, New York.

Mindlin, R.D., 1949. Compliance of elastic bodies in contact. ASME Journal of Applied Mechanics 16 (3) 259–268.

Mayrhoffer, K., Fischer, F.D., 1994. Analytical solutions and numerical algorithm for the Gauss's hypergeometric function ${}_2F_1(a, b; c; d)$. Zeitschrift fur Angewandte Mathematik und Mechanik 74 (7), 265–273.

Pouyet, J., Lataillaze, J.L., 1979. Elastic strains of a transversely isotropic half-space caused by a rigid sphere. Letters in Applied and Engineering Sciences 17, 765–772.

Shield, R.T., 1951. Notes on problems in hexagonal aelotropic materials. Proceedings of the Cambridge Philosophical Society 47, 401–409.

Sneddon, I.N., 1972. The Use of Integral Transforms. McGraw-Hill, New York.

Sveklo, V.A., 1964. Boussinesq type problems for the anisotropic half space. Journal of Applied Mathematics and Mechanics (PMM) 28, 1099–1105.

Sveklo, V.A., 1970. The action of a stamp on an elastic anisotropic half-space. Journal of Applied Mathematics and Mechanics (PMM) 34 (1), 165–171.

Ting, T.C.T., 1996. Anisotropic Elasticity. Oxford University Press.

Vorovich, I.I., Alexandrov, V.M., Babeshko, V.A., 1974. Non-Classical Mixed Boundary Value Problems of the Theory of Elasticity. Nauka, Moscow (in Russian).

Willis, J.R., 1966. Hertzian contact of anisotropic bodies. Journal of the Mechanics and Physics of Solids 15, 163–176.